ON THE SECOND METHOD OF LIAPUNOV

(KO VTOROI METODE LIAPUNOVA)

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S.K. PERSIDSKII
(Alma-Ata)

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Let us consider the system of differential equations

$$\frac{dx_s}{dt} = f_s(t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{1}$$

where the f_s are real functions given in some region

$$(h)\ t \geqslant 0, \qquad \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} \leqslant R$$

are continuous in t and satisfy the Cauchy condition relative to x_1, $\ldots$, x_n; furthermore, $f_s(t, 0, 0, \ldots, 0) \equiv 0$.

Suppose that $\phi(\tau)$ is a real continuous function when $\tau \geqslant 0$, which has a continuous derivative that satisfies the following two conditions:

$$(1) \quad \varphi(\tau) = 1 \text{ when } 0 \leqslant \tau \leqslant R_0 < R_1 < R, \qquad (2) \quad \varphi(\tau) = 0 \quad \text{when} \quad \tau \geqslant R_1$$

Let us introduce the functions

$$F_s(t, x_1, \ldots, x_n) = f_s(t, x_1, \ldots, x_n)\, \varphi(\|x\|) \qquad (s = 1, \ldots, n)$$

where for the sake of definiteness we assume that $f_s(t, x_1, \ldots, x_n) \equiv 0$ when $\|x\| > R$. It is not difficult to see [1] that the functions F_s are continuous in t and satisfy the Cauchy condition relative to $x_1, \ldots, x_n$ in the region

$$(H) \quad t \geqslant 0, \qquad \|x\| < \infty$$

Let us consider the system of differential equations

$$\frac{dx_s}{dt} = F_s(t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{2}$$

It is obvious that through each point $(t_0, x_{10}, \ldots, x_{n0})$ of the region H there passes a unique solution

$$x_s = x_s(t, t_0, x_{10}, \ldots, x_{n0}) \qquad (s = 1, \ldots, n) \tag{3}$$

of the system (2). It is well known [1,2] that the relation (3) can be solved for x_{s0} in H and that

$$x_{s0} = x_s(t_0, t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{4}$$

In the region

$$(h_0) \quad t \geqslant 0, \qquad \|x\| \leqslant R_0$$

the solution of the system (2) coincides with the corresponding solutions of the system (1). Therefore, the zero solutions of the indicated systems of equations are equivalent to each other with respect to stability in the Liapunov sense [3].

Definition 1. Suppose that in some region

$$(g) \quad t \geqslant 0, \quad \|x\| \leqslant r \qquad (r \leqslant R_0)$$

there is defined a continuous single-valued function $V(t, x_1, \ldots, x_n)$ which is of a definite sign and positive for any fixed value $t \geqslant 0$. Let us assume that the function $V(t, x_1, \ldots, x_n)$ is such that there exists some real constant $a > 0$ ($a < r$) such that for every initial value $t = t_0 \geqslant 0$ and for every given $\epsilon > 0$ there exists a value $t = T(\epsilon, t_0) > t_0$ such that in the plane $t = T$ of the region g one can always connect the point $O(T, 0, 0, \ldots, 0)$ with the surface $\|x\| = a$ by means of a continuous curve Γ at all of whose points

$$V(T, x_1, \ldots, x_n) < \varepsilon \tag{5}$$

Then we shall say that the given function $V(t, x_1, \ldots, x_n)$ is a positive-weakly definite function.

Definition 2. Let $V(t, x_1, \ldots, x_n)$ be a positive-definite function in the region g, and let $\eta > 0$ be an arbitrary given number. We denote by $D(\eta)$ the set of all those points of the region g at which the following inequality is satisfied:

$$V(t, x_1, \ldots, x_n) \leqslant \eta \tag{6}$$

Let $t = t^* \geqslant 0$. The intersection of the set $D(\eta)$ with the plane $t = t^*$ in the region g we shall denote by $\sigma(\eta, t^*)$.

If for every sufficiently small $\eta > 0$ the maximum norm of the points $(t^*, x_1, \ldots, x_n) \in \sigma(\eta, t^*)$ satisfies the condition

$$\max \| x \| \to 0 \text{ when } t^* \to \infty$$

then we shall say that the given function $V(t, x_1, \ldots, x_n)$ is positive-strongly definite.

For example, let

$$V(t, x_1, \ldots, x_n) = \sum_{s, k=1}^{n} a_{sk}(t) x_s x_k \qquad (a_{sk} = a_{ks}) \tag{7}$$

be a quadratic form which is positive-definite for any fixed $t \geqslant 0$. We shall denote by $\lambda_1(t)$ the smallest root of the equation $\det | a_{sk}(t) - \lambda \delta_{sk} | = 0$.

It is easy to show [4] that the quadratic form (7) will be positive-strongly definite if and only if

$$\lim \lambda_1(t) = \infty \text{ when } t \to \infty$$

In order that the considered quadratic form be positive-weakly definite it is necessary and sufficient that

$$\lim \lambda_1(t) = 0 \text{ when } t \to \infty$$

Let $t_0 \geqslant 0$, and let us denote by

$$x_s = x_s(t, t_0, x_{10}, \ldots, x_{n0}) \qquad (s = 1, \ldots, n) \tag{8}$$

the solution of (2) which passes for $t = t_0$ through an arbitrary given point $(t_0, x_{10}, \ldots, x_{n0})$ of the region H. Let us solve (8) for x_{s0} (in accordance with Formula (4)) and set

$$V(t, x_1, \ldots, x_n) = \sum_{s=1}^{n} x_s^2(t_0, t, x_1, \ldots, x_n) \tag{9}$$

It is obvious that the considered function $V(t, x_1, \ldots, x_n)$ is positive-definite for every fixed value of $t \geqslant 0$, while in the region H

$$V'(t, x_1, \ldots, x_n) \equiv 0 \tag{10}$$

because of (2).

1. We shall show that if the zero solution of the system (1) is unstable then the function

$$V_0(t, x_1, \ldots, x_n) = \sum_{s=1}^{n} x_s^2(0, t, x_1, \ldots, x_n) \tag{11}$$

defined by the relation (9), with $t_0 = 0$, will be positive-weakly definite in the region h_0.

Indeed, if the zero solution of the system (1) and hence of the system (2), is unstable then there exists a real constant $a > 0$ ($a < R_0$) such that among the possible solutions of the system (2)

$$x_s = u_s(t, x_{10}, \ldots, x_{n0}) \qquad (s = 1, \ldots, n) \tag{12}$$

which satisfy for $t = 0$ the initial conditions

$$\|x_0\| = \tfrac{1}{2}\sqrt{3\varepsilon} < \alpha \tag{13}$$

there exists at least one solution

$$x_s = u_s(t, \xi_{10}, \ldots, \xi_{n0}) \qquad (s = 1, \ldots, n) \qquad \left(\|\xi_0\| = \tfrac{1}{2}\sqrt{3\varepsilon}\right) \tag{14}$$

and one value $t = T(\epsilon) > 0$ such that an arbitrary solution (12) will satisfy the inequality

$$\|u(t, x_{10}, \ldots, x_{n0})\| < \alpha \tag{15}$$

for all values $t \in (0, T)$, and the solution (14), with $t = T(\epsilon)$, will satisfy the condition

$$\|u(T, \xi_{10}, \ldots, \xi_{n0})\| = \alpha \tag{16}$$

Furthermore, if $t = t_0 \geqslant 0$ is an arbitrarily chosen fixed value of t, then, in view of the continuity of the solution of the system (2) with respect to the initial values, one may always assume that the positive number ϵ in (13) has been chosen so small that the above mentioned $T = T(\epsilon)$ is greater than t_0. It is obvious that the geometric locus of the points which for $t \geqslant 0$ lie on the integral curves (12) form the surface

$$V_0(t, x_1, \ldots, x_n) = \tfrac{3}{4}\varepsilon \tag{17}$$

It follows from (16) that there exists a point $M(T, x_1', \ldots, x_n')$ on the indicated surface, with $t = T$, at which $\|x'\| = a$.

When $t = T(\epsilon)$, the surface (17) determines some connected closed set $\gamma(3\epsilon/4, T)$ for which the points of the surface (17), with $t = T$, are boundary points and at all of whose points it is true that

$$V_0(T, x_1, \ldots, x_n) \leqslant \tfrac{3}{4}\varepsilon \tag{18}$$

Here, the point $O(T, 0, 0, \ldots, 0)$ is an interior point of the set

$\gamma(3\epsilon/4,\ T)$ while the point $M(T,\ x_1',\ \ldots,\ x_n')$, at which $\|x'\| = a$, is a boundary point of this set. From (18) it follows that the indicated points O and M can be connected in the plane $t = T\ (\epsilon)$ by some continuous curve Γ at all of whose points the following inequality holds:

$$V_0(T, x_1, \ldots, x_n) \leqslant \frac{3}{4}\varepsilon < \varepsilon \tag{19}$$

Hence, in the case under consideration, the function V_0 is positive-weakly definite, and it follows from (10) that $V' \equiv 0$ in the region h_0 because of the system of equations (1).

2. Suppose that the zero solution of the system (1), and hence also of the system (2), is asymptotically uniformly stable with respect to the coordinates $x_{10},\ \ldots,\ x_{n0}$. We shall show that the above-considered function $V_0(t,\ x_1,\ \ldots,\ x_n)$ is positive-strongly definite in the region h.

Let us denote by

$$x_s = u_s(t, x_{10}, \ldots, x_{n0}) \qquad (s = 1, \ldots, n) \tag{20}$$

the solutions of the system (2) which satisfy for $t = 0$ the initial conditions

$$\|x_0\| \leqslant \eta \tag{21}$$

where $\eta > 0$ is an arbitrary given sufficiently small number such that the solution (20) satisfies the following condition:

$$\|u(t, x_{10}, \ldots, x_{n0})\| \to 0 \qquad \text{when } t \to \infty \tag{22}$$

uniformly in the $x_{10},\ \ldots,\ x_{n0}$, which are connected by the relation (21).

Let $t = t^* \geqslant 0$. The intersection of the set of all those points where $V_0(t,\ x_1,\ \ldots,\ x_n) \leqslant \eta$ with the plane $t = t^*$ will be denoted by $\sigma(\eta,\ t^*)$. It is obvious that this set is the geometric locus of the points which for $t = t^*$ lie on the integral curves (20). Since the integral curves (20) satisfy condition (22) uniformly in the $x_{10},\ \ldots,\ x_{n0}$, which are connected by the relation (21), it is obvious that the maximum of the norms of the points $(t^*,\ x_1,\ \ldots,\ x_n) \in \sigma(\eta,\ t^*)$ satisfies the condition

$$\max\|x\| \to 0 \qquad \text{when } t^* \to \infty \tag{23}$$

Furthermore, it is well known [1,2] that in the case of stability of the zero solution of the system (1), the function $V_0(t,\ x_1,\ \ldots,\ x_n)$ is positive-definite in the region h_0.

Thus in the case under consideration the function $V_0(t,\ x_1,\ \ldots,\ x_n)$ is indeed positive-strongly definite in the region h_0.

Theorem 1. In order that the zero solution of the system (1) may be unstable it is necessary and sufficient that in some region

$$(g) \quad t \geqslant 0, \qquad \|x\| \leqslant r \leqslant R_0$$

there exist a positive-weakly definite function $V(t, x_1, \ldots, x_n)$ such that $V'(t, x_1, \ldots, x_n) \geqslant 0$ on the basis of Equations (1).

Sufficiency. Let us assume that the zero solution of the system (1) is stable. Then for every given $t_0 \geqslant 0$ and for the real constant $a > 0$ which appears in the definition of the weakly definite function V, there exists a number $\delta = \delta(a, t_0) > 0$ such that the integral curves

$$x_s = x_s(t, t_0, x_{10}, \ldots, x_{n0}) \qquad (s = 1, \ldots, n) \tag{24}$$

of the system (1) which pass for $t = t_0$ through the points of the spherical surface

$$\|x_0\| = \delta \tag{25}$$

will satisfy the inequality

$$\|x(t, t_0, x_{10}, \ldots, x_{n0})\| < \alpha \tag{26}$$

for all $t \geqslant t_0$.

Let $l = \min V(t_0, x_1, \ldots, x_n)$ if $\|x\| < \delta$. Then, in view of the fact that $V' \geqslant 0$ at all points which for $t \geqslant t_0$ lie on the integral curves (24), we shall have

$$V(t, x_1, \ldots, x_n) \geqslant l \tag{27}$$

By the hypotheses of the theorem, the function V is such that for every given $\epsilon > 0$ $(\epsilon < l)$ there exists a value $t = T(\epsilon, t_0) > t_0$ such that in the plane $t = T$ of the region g the point $O(T, 0, 0, \ldots, 0)$ can always be connected with at least one point $M(T, x_1', \ldots, x_n')$ lying on the surface $\|x\| = a$ by means of a continuous curve Γ at all of whose points the next inequality holds:

$$V(T, x_1, \ldots, x_n) < \varepsilon < l \tag{28}$$

The integral curves (24) form in the plane $t = T$ some surface $S(\delta, t_0, T)$ at all of whose points condition (27) is satisfied, and which is homeomorphic to the spherical surface $\|x_0\| = \delta(a, t_0)$ of the plane $t = t_0$ of the region g. The surface $S(\delta, t_0, T)$ determines in the plane $t = T$ some closed set $\gamma(\delta, t_0, T)$ which contains the points of the surface $S(\delta, t_0, T)$ in its boundary; furthermore, the point $O(T, 0, 0, \ldots, 0) \in \gamma(\delta, t_0, T)$ is an interior point of this set.

From this, and on the basis of (26), we conclude that the above-indicated curve Γ must have at least one point in common with the surface $S(\delta, t_0, T)$, and that at this common point the contradictory conditions (27) and (28) must both be satisfied. The obtained contradiction establishes that the assumption on the stability of the zero solution of the system (1) is not valid.

Necessity. In Section 1 it was shown that the function $V_0(t, x_1, \ldots, x_n)$ defined by the relation (11) is (in case of instability of the zero solution of the system (1)) a positive-weakly definite function in the region h_0 and that $V' \equiv 0$ in this region in view of the system (1). It is not difficult to see that in the given case the function $W(t, x_1, \ldots, x_n) = V_0(t, x_1, \ldots, x_n)(2 - e^{-t})$ will also be a positive-weakly definite function in the region h_0 and that $W' > 0$ when $\|x\| > 0$, in view of the system (1).

This completes the proof of the theorem.

As an example of the application of Theorem 1 we shall investigate the stability of the zero solution of the differential equation

$$\frac{dx}{dt} = -2xyz, \qquad \frac{dy}{dt} = \frac{y}{t+2} + x^2 z, \qquad \frac{dz}{dt} = \frac{z}{t+2} + x^2 y \tag{29}$$

It is not difficult to see that the function

$$V(t, x, y, z,) = x^2 + y^2 + z^2 + 2\,\frac{t+1}{t+2}\,zy$$

is positive-weakly definite and that in view of the system (29) we have

$$V'(t, x, y, z) = \frac{2}{t+2}\left[z^2 + y^2 + \frac{2t+3}{t+2}\,zy + x^2 (t+1)(z^2 + y^2)\right] \geqslant 0$$

Hence, the zero solution of the system of equations under consideration is not stable.

Theorem 2. In order that the zero solution of the system (1) may be asymptotically uniformly stable with respect to the coordinates $x_{10}, \ldots, x_{n0}$, it is necessary and sufficient that in some region

$$(g) \qquad t \geqslant 0, \qquad \|x\| \leqslant r \leqslant R_0$$

there exist a positive-strongly definite function $V(t, x_1, \ldots, x_n)$ such that $V'(t, x_1, \ldots, x_n) \leqslant 0$ in view of Equations (1).

Sufficiency. Let $r_0 > 0 (r_0 < r)$ be given arbitrarily. The function $V(t, x_1, \ldots, x_n)$ is positive-definite. Therefore there exists a sufficiently small number $\eta_0 > 0$ such that the points of the set $D(\eta_0)$, which consists of all those points $(t, x_1, \ldots, x_n) \in g$ at which

$$V(t, x_1, \dots, x_n) \leqslant \eta_0 \tag{30}$$

will belong to the region g_0:

$$(g_0) \qquad t \geqslant 0, \qquad \|x\| \leqslant r_0$$

Furthermore, it is obvious that for every $\eta \leqslant \eta_0$ ($\eta > 0$) all the points of the $D(\eta)$ will also belong to the region g_0. Let $\eta_0 > 0$ ($\eta_1 < \eta_0$) be an arbitrarily chosen number such that the maximum of the norms of the points $(t^*, x_1, \dots, x_n) \in \sigma(\eta_1, t^*)$, where $\sigma(\eta, t^*)$ is the intersection of the set $D(\eta_1)$ with the plane $t = t^* \geqslant 0$, satisfies the condition

$$\max \|x\| \to 0 \quad \text{when } t^* \to \infty \tag{31}$$

(such a choice of the number η_1, obviously, is possible because of the fact that $V(t, x_1, \dots, x_n)$ is a positive-strongly definite function).

Let $t_0 \geqslant 0$ be an arbitrarily chosen value of t. We select a number $\delta = \delta(t_0, \eta_1) > 0$ ($\delta < r_0$) such that

$$V(t_0, x_1, \dots, x_n) < \eta_1 \tag{32}$$

whenever $\|x\| \leqslant \delta(t_0, \eta_1)$. Since in the given case the zero solution of the system (1) is stable by a theorem of Liapunov [3], we shall assume that the indicated number $\delta(t_0, \eta_1)$ is chosen in such a way that all solutions

$$x_s = x_s(t, t_0, x_{10}, \dots, x_{n0}) \qquad (s = 1, 2, \dots, n) \tag{33}$$

of the system (1) which satisfy, for $t = t_0$, the initial conditions

$$\|x_0\| \leqslant \delta(t_0, \eta_1) \tag{34}$$

will lie in the region g_0 for all $t \geqslant t_0$. By the hypothesis of the theorem, $V'(t, x_1, \dots, x_n) \leqslant 0$. Therefore, on the basis of (34) and (32), we can conclude that the set of all points $(t, x_1, \dots, x_n)$ which lie for $t \geqslant t_0$ on the integral lines (33) belong to the set $D(\eta_1)$. But then it follows from (31) that the solutions (33) of the system (1) satisfy the condition

$$\|x(t, t_0, x_{10}, \dots, x_{n0})\| \to 0 \quad \text{when } t \to \infty \tag{35}$$

uniformly in $x_{10}, \dots, x_{n0}$, which belong to the region

$$t = t_0, \qquad \|x_0\| \leqslant \delta(t_0, \eta_1)$$

Necessity. In Section 2 it was shown that in the case of an asymptotic zero solution of the system (1), which is uniform in the coordinates $x_{10}, \ldots, x_{n0}$, the function $V(t, x_1, \ldots, x_n)$ defined by the relation (11) will be positive-strongly definite in the region h_0, while in the same region $V' \equiv 0$, in view of the system (1). It is not difficult to see that in this case, also, the functions

$$U(t, x_1, \ldots, x_n) = V_0(t, x_1, \ldots, x_n)(1 + e^{-t})$$

are positive-strongly definite in the region h_0, while in the same region $U' < 0$ if $\|x\| > 0$. This completes the proof of the theorem.

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